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THE RELATIVE ISOMORPHISM THEOREM FOR BERNOULLI FLOWS

BY

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ABSTRACT

In this paper we extend the work of Thouvenot and others on Bernoulli splitting of ergodic transformations to ergodic flows of finite entropy. We prove that if $\mathcal{A}$ is a factor of a flow S, where S_1 is ergodic and $\mathcal{A}$ has a Bernoulli complement in S_1, then $\mathcal{A}$ has a Bernoulli complement in S. Consequently, Bernoulli splitting for flows is stable under taking intermediate factors and certain $\bar{d}$ limits. In addition it follows that the property of isomorphism with a Bernoulli $\times$ zero entropy flow is similarly stable.

1. Introduction

In the ground-breaking work of [4] and subsequent papers, D. S. Ornstein introduced the finitely determined property which characterized those processes isomorphic to a Bernoulli shift. It was shown that Bernoulli shifts are finitely determined, and that any two finitely determined processes of the same entropy are isomorphic. This work then led to the solution of a number of other outstanding problems about Bernoulli shifts, such as the factor problem and the root problem.

A deep and important generalization of this work was given by J.-P. Thouvenot [9] who obtained conditional versions of these theorems. In his work, he introduced the relatively finitely determined property which characterized those factors of an ergodic automorphism which admit a Bernoulli complement.

On the other hand, Ornstein adapted the proof of his isomorphism theorem to obtain the isomorphism theorem for Bernoulli flows [5]. In this paper, we unite these two lines of research and develop a "relativized" theory of Bernoulli flows. The main theorem (Theorem 2) is an isomorphism theorem analogous to those of [5] and [9]. From this theorem we quickly generalize a number of the results of Thouvenot from the discrete case. It should be noted here that the main result

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and its consequences extend to the case of $\mathbf{R}^n$-flows. These generalizations may be obtained by routine changes of notation in the present arguments and by use of the appropriate generalizations of theorems such as the Shannon–MacMillan theorem, the ergodic theorem and the Rokhlin lemma. The essential ideas are present in the one-dimensional case, however, so we have restricted our attention to this case in the hope of enhancing the clarity of the exposition.

All probability spaces $(X, \mathcal{F}, \mu)$ in this paper will be assumed to be Lebesgue spaces. All partitions of a probability space will be assumed finite and ordered. If $P = \{P_1, \cdots, P_k\}$ is a partition of $(X, \mathcal{F}, \mu)$ we let dist(P) denote the ordered k-tuple $\{\mu(P_k), \cdots, \mu(P_k)\}$. If E is a set of positive measure, we let $\text{dist}_E(P)$ denote $\{\mu(P_i \cap E)/\mu(E)\}_{i=1}^k$. If Q is another partition with k elements, we let

$$|\text{dist}(P) - \text{dist}(Q)| = \sum_{i=1}^{k} |\mu(P_i) - \mu(Q_i)|.$$

If Q is also a partition of $(X, \mathcal{F}, \mu)$, we write $|P - Q| = \Sigma_{i=1}^k \mu(P_i \triangle Q_i)$. This gives rise to what is called the partition metric. Dropping the requirement that $|P| = |Q|$, we say P refines Q and write $P \supset Q$ if every atom of Q is a union of atoms of P. We write $P \supset^{\varepsilon} Q$ for $\varepsilon > 0$ if there exists a partition Q' such that $|Q - Q'| < \varepsilon$ and $P \supset Q'$. In this situation it is clear that Q' determines and is determined by a function $\pi : P \to Q'$ such that $\pi(P_i) \supset P_i$, $i = 1, 2, \cdots, k$, and we write $Q' = \pi(P)$ and call π a code from P to Q'.

Given a measure preserving transformation T on a space $(X, \mathcal{F}, \mu)$ and a partition P of X, we let (T, P) denote the corresponding process. The process (T, P) determines a factor of T, namely $(P)_T = \bigvee_{-\infty}^{\infty} T^i P$, and we write $(T, (P)_T)$ to indicate the action of T restricted to this factor. A process (T, P) is ergodic if $(T, (P)_T)$ is ergodic. If $(P)_T = \mathcal{F}$, we say P is a generator for T. Given two processes (T, P) and $(\bar{T}, \bar{P})$ we write $(T, P) \sim (\bar{T}, \bar{P})$ if $(\forall n)$ dist $\bigvee_{i=1}^n T^{-i} P =$ dist $\bigvee_{i=1}^n \bar{T}^{-1} \bar{P}$.

For the reader's convenience we include a summary of the work of Thouvenot referred to above. We begin by describing the relative $\bar{d}$ metric. Let $(T, P \vee Q)$ be an ergodic process on $(X, \mathcal{F}, \mu)$ and $(T', P' \vee Q')$ an ergodic process on $(X', \mathcal{F}', \mu')$ such that $|P| = |P'|$ and $(T, Q) \sim (T', Q')$.

For each $n = 1, 2, 3$ we set

$$\bar{d}^n_{Q,Q'}[(T, P \vee Q), (T', P' \vee Q')] = \inf_{\psi} \frac{1}{n} \sum_{i=0}^{n-1} |\psi(T^i P) - (T')^i P'|$$

where the infimum is taken over all isomorphisms $\psi : X \to X'$ such that $(\forall m)$ $\psi(\bigvee_{i=0}^{m-1} T^{-1} Q) = \bigvee_{i=0}^{m-1} (T')^{-1} Q'$. We then put

$$\bar{d}_{Q,Q'}[(T, P \vee Q), (T', P' \vee Q')] = \sup_n \bar{d}^n_{Q,Q'}[(T, P \vee Q), (T', P' \vee Q')]$$

which is easily shown to equal the limit.

This definition, though apparently stronger, is in fact equivalent to the usual definition of $\bar{d}_{Q,Q'}$ (see [9]), but it will be convenient for us to use this one.

Recall that an automorphism T of $(X, \mathcal{F}, \mu)$ is called a Bernoulli shift if there is a generator B for T such that (T, B) is an independent process. In other words, $(\forall N)$ $T^n B \perp \bigvee_{i=0}^{n-1} T^{-i}B$. We say a factor $\mathcal{C} \subset \mathcal{F}$ of T has a Bernoulli complement (in $(T, \mathcal{F})$) if there is a factor $\mathcal{B} \subset \mathcal{F}$ of T such that $\mathcal{B} \perp \mathcal{C}$, $\mathcal{B} \vee \mathcal{C} = \mathcal{F}$, and T is a Bernoulli shift on $(X, \mathcal{B}, \mu)$. We now state the property that was used by Thouvenot to characterize factors of this type.

Definition. An ergodic process $(T, P \vee Q)$ is called Q-relatively finitely determined (briefly: Q-rel.f.d.) if $(\forall \varepsilon > 0)$ $(\exists \delta > 0,\ u \in \mathbf{N})$ such that for all ergodic processes $(T', P' \vee Q')$ with $(T', Q') \sim (T, Q)$, the conditions

(1) $|h(T, P \vee Q) - h(T', P' \vee Q')| < \delta$ and

(2) $|\text{dist} \bigvee_{i=0}^{u} T^{-i}(P \vee Q) - \text{dist} \bigvee_{i=0}^{u} (T')^{-i}(P' \vee Q')| < \delta$

imply $\bar{d}_{Q,Q'}[(T, P \vee Q), (T', P' \vee Q')] < \varepsilon$.

We have the following important theorems.

Theorem A. *If $(T, B \vee Q)$ is an ergodic process on $(X, \mathcal{F}, \mu)$ such that $(B)_T \perp (Q)_T$, $(B)_T \vee (Q)_T = \mathcal{F}$, and (T, B) is an independent process, then $(T, B \vee Q)$ is* Q-rel.f.d.

Theorem B. *Let $(T, P \vee Q)$ be an ergodic process on $(X, \mathcal{F}, \mu)$ which is* Q-rel.f.d. *and for which $(P \vee Q)_T = \mathcal{F}$. Let $(T', P' \vee Q')$ on $(X', \mathcal{F}', \mu')$ satisfy the same description and in addition satisfy*

(1) $h(T') = h(T)$,

(2) $(T, Q) \sim (T', Q')$.

Then T is isomorphic to T' relative to the common factors (T, Q) and (T', Q'). In other words, there is an isomorphism $\psi : (X, \mathcal{F}, \mu) \to (X', \mathcal{F}', \mu')$ such that $\psi T(x) = T'\psi(x)$ (a.e. x) *and $\psi(Q) = Q'$.*

We then immediately deduce a converse to Theorem A.

Theorem A′. *If $(T, P \vee Q)$ is a* Q-rel.f.d. *process on $(X, \mathcal{F}, \mu)$ where $(P \vee Q)_T = \mathcal{F}$, then $(Q)_T$ has a Bernoulli complement in $(P \vee Q)_T$.*

The following results, excepting Theorem D, are also due to Thouvenot [9], [10], or Thouvenot and Shields [8] and will be used and generalized in this paper.

Theorem D does not appear in these papers, but it can be proved exactly as is the analogous theorem in the non-relative case.

THEOREM C. *If* $(T, P \vee Q)$ *is a* Q-rel.f.d. *process on* $(X, \mathcal{F}, \mu)$ *and* R *is another partition of* X, *then* $(T, R \vee Q)$ *is a* Q-rel.f.d. *process.*

THEOREM D. *If, for some* $n \in \mathbf{N}$, $(T^n, P \vee Q)$ *is a* Q-rel.f.d. *process on* $(X, \mathcal{F}, \mu)$ *where* $(P \vee Q)_{T^n} = \mathcal{F}$ *and* $(Q)_{T^n} = (Q)_T$, *then* $(T, P \vee Q)$ *is* Q-rel.f.d.

THEOREM E. *If* $\{(T_i, P_i \vee Q_i)\}_{i=1}^{\infty}$ *is a sequence of* Q_i-rel.f.d. *processes which converge in* $\bar{d}$ *to a process* $(T, P \vee Q)$ *and each* (T_i, Q_i) *is isomorphic to* (T, Q), *then* $(T, P \vee Q)$ *is* Q-rel.f.d.

THEOREM F. (a) *If* T *is an automorphism isomorphic to the direct product of a Bernoulli shift and an ergodic automorphism of zero entropy, then every factor of* T *decomposes in the same way.*

(b) *If* $\{(T_i, P_i)\}_{i=1}^{\infty}$ *is a sequence of processes converging in* $\bar{d}$ *to* (T, P) *where each* T_i *is as in* (a), *then* $(T, (P)_T)$ *decomposes as in* (a).

We now turn to preliminaries for our investigation of flows.

Given a flow $S = \{S_t\}_{t \in R}$ and a partition P of $(X, \mathcal{F}, \mu)$, we let (S, P) denote the corresponding (continuous) process, and $(P)_S$ the factor $\bigvee_{t \in R} S_t P$ determined by P. As in the discrete case, we write $(S, (P)_S)$ to denote the action of S restricted to this factor. Given two processes (S, P) and (S', P') we write $(S, P) \sim (S', P')$ if $(\forall n)$ $(\forall t_1 < t_2 < \cdots < t_n \in \mathbf{R})$

$$\operatorname{dist} \bigvee_{i=1}^{n} S_{t_i}(P) = \operatorname{dist} \bigvee_{i=1}^{n} S_{t_i}'(P').$$

We define the entropy of a flow S on $(X, \mathcal{F}, \mu)$ to be $h(S_1)$, and we define the entropy of a process (S, P) to be $h(S_1, (P)_S)$. All flows in this paper are assumed to be of finite entropy.

Let (S, Q) and $(\bar{S}, \bar{Q})$ be processes on $(X, \mathcal{F}, \mu)$ and $(\bar{X}, \bar{\mathcal{F}}, \bar{\mu})$ respectively, such that $(S, Q) \sim (\bar{S}, \bar{Q})$. We say that S and $\bar{S}$ are isomorphic relative to (S, Q) and $(\bar{S}, \bar{Q})$ if there is an isomorphism $\psi : X \to \bar{X}$ such that $(\forall t \in \mathbf{R})$ $\psi(S_t x) = \bar{S}_t \psi(x)$ for all x outside a flow-invariant null set, and $(\forall t \in \mathbf{R})$ $\psi(S_t Q) = \bar{S}_t \bar{Q}$.

We now define a relative $\bar{d}$-metric for such processes as we did for discrete processes. For $n = 1, 2, 3, \cdots$ put

$$\bar{d}^{\,n}_{\bar{Q}, Q}[(\bar{S}, \bar{P} \vee \bar{Q}), (S, P \vee Q)] = \inf_{\psi} \frac{1}{n} \int_0^n |\psi(\bar{S}_t \bar{P}) - S_t P| \, dt$$

where the infimum is taken over all isomorphisms $\psi : \bar{X} \to X$ such that $(\forall t \in \mathbf{R})$ $\psi(\bar{S}_t\bar{Q}) = S_tQ$. We then set

$$\bar{d}_{\bar{Q},Q}[(\bar{S}, \bar{P} \vee \bar{Q}), (S, P \vee Q)] = \sup_n \bar{d}^{\,n}_{\bar{Q},Q}[(\bar{S}, \bar{P} \vee \bar{Q}), (S, P \vee Q)].$$

We will need two facts about this metric, the first of which is easily verified, and the second of which is proved as in the non-relative case (cf. [3] or [6]).

FACT 1. If $\bar{d}_{\bar{Q},Q}[(\bar{S}, \bar{P} \vee \bar{Q}), (S, P \vee Q)] = \alpha$, then

$$(\forall t_0 \in \mathbf{R}) \quad \bar{d}_{\bar{Q},Q}[(\bar{S}_{t_0}, \bar{P} \vee \bar{Q}), (S_{t_0}, P \vee Q)] \leqq \alpha.$$

FACT 2. If $\bar{d}_{\bar{Q},Q}[(\bar{S}, \bar{P} \vee \bar{Q}), (S, P \vee Q)] = 0$, then $(\bar{S}, (\bar{P} \vee \bar{Q})_{\bar{S}})$ and $(S, (P \vee Q)_S)$ are isomorphic relative to $(\bar{S}, \bar{Q})$ and (S, Q).

We will use the following version of the Rokhlin lemma (cf. [1] and [2]).

THEOREM G. *Let S be an ergodic flow on $(X, \mathcal{F}, \mu)$, and let P be a partition of X. Then $(\forall \varepsilon > 0)$ $(\forall L > 0)$ $(\exists F \in \mathcal{F})$ so that the $\{S_tF, t \in [0, L]\}$ are disjoint and $\bigcup_{t=0}^{L} S_tF \in \mathcal{F}$ with $\mu(\bigcup_{t=0}^{L} S_tF) > 1 - \varepsilon$. Furthermore, if $\pi : \bigcup_{t=0}^{L} S_tF \to F$ is the obvious projection and μ_F is the normalization of $\mu \circ \pi^{-1}$ then we have*

$$\left| \operatorname*{dist}_{(F,\mu_F)}(P) - \operatorname*{dist}_{(X,\mu)}(P) \right| < \varepsilon,$$

and μ restricted to $T = \bigcup_{t=0}^{L} S_tF$ is equal to the product of μ_F and the Lebesgue measure on $[0, L]$, appropriately normalized. In this situation we will write $T = \bigcup_{t=0}^{L} S_tF$ and say (T, F, μ_F) is a Rokhlin tower of height L in $(S, X, \mathcal{F}, \mu)$.

A flow S on $(X, \mathcal{F}, \mu)$ is called Bernoulli if S_1 is a Bernoulli shift of $(X, \mathcal{F}, \mu)$.

The following theorems are proven by Ornstein [5]:

THEOREM H. *All Bernoulli flows of a given entropy are isomorphic.*

THEOREM J. *There exists a flow such that $(\forall t \in \mathbf{R})$ S_t is a Bernoulli shift.*

These theorems imply of course, that if a flow S has S_1 a Bernoulli shift, then $(\forall t \in \mathbf{R})$ S_t is a Bernoulli shift.

We say a factor $\mathcal{C} \subset \mathcal{F}$ of a flow S on $(X, \mathcal{F}, \mu)$ has a Bernoulli complement (in $(S, \mathcal{F})$) if there exists a factor $\mathcal{B} \subset \mathcal{F}$ of S such that $\mathcal{B} \perp \mathcal{C}$, $\mathcal{B} \vee \mathcal{C} = \mathcal{F}$ and $(S, \mathcal{B})$ is a Bernoulli flow. Our main result (Theorem 2′) will say that if S_1 is ergodic, and $\mathcal{C}$ has a Bernoulli complement in $(S_1, \mathcal{F})$, then it has a Bernoulli complement in $(S, \mathcal{F})$.

2. The relative isomorphism theorem

The method we use for handling continuous names is an adaptation of that of Lind [2]. We will use the following notation. For a measurable function $f: X \times \mathbf{R} \to \mathbf{R}^d$, $n \in \mathbf{N}$, and a measurable set $E \subset X$, we put

$$\|f\|_E = \frac{1}{\mu(E)} \int_E |f(x,0)|\, d\mu(x),$$

$$\|f(x)\|_n = \frac{1}{n} \int_0^n |f(x,t)|\, dt,$$

and

$$\|f\|_{E,n} = \frac{1}{n} \int_0^n \frac{1}{\mu(E)} \int_E |f(x,t)|\, d\mu(x) dt$$

where the norms under the integral signs denote the L^1-norm on $\mathbf{R}^d$.

When a partition is denoted by a capital letter, (say P), then the corresponding lower case letter will be used to indicate the naming function $p(x,t): X \times \mathbf{R} \to P$ given by $p(x,t) = P_i \Leftrightarrow S_t x \in P_i$. If P is a finite partition of X, and $p(x,t)$ is the associated naming function, then by identifying P with the standard basis of $\mathbf{R}^P$, we regard p as a map p: $X \times \mathbf{R} \to \mathbf{R}^P$, and for $J \in \mathbf{N}$, set

$$P_J(x,t) = \frac{J}{2} \int_{-1/J}^{1/J} p(x, t+s) ds.$$

We note that, by the Wiener local ergodic theorem [11], $\lim_{J\to\infty} P_J(x,0) = p(x,0)$ a.e. Hence $\lim_{J\to\infty} \|p - p_J\|_X = 0$. For $N \in \mathbf{N}$, we let $\psi_N p(x,t) = p(x, i/N)$ where $t \in [i/N, (i+1)/N)$. We call $\psi_N p(x,t)$ the N-filled version of $p(x,t)$. The following lemmas set the stage for the main copying theorem. They tell us that in order to obtain a continuous relative $\bar{d}$-approximation of one continuous process in another, it is sufficient (roughly speaking) to obtain a discrete relative $\bar{d}$-approximation provided the new continuous names are well approximated by their filled versions.

In the following, $p(x,t)$ is the naming function arising from a process (S, P).

LEMMA 1.

$$(\forall \varepsilon > 0)(\exists \bar{N})(\forall N > \bar{N} \text{ and } k \in \mathbf{N}) \quad \|p - \psi_N p\|_{X,k/N} < \varepsilon.$$

PROOF. Take $J \in \mathbf{N}$ so that $\|p - p_J\|_X < \varepsilon/3$. Then $(\forall k)$

$$\|p - \psi_N p\|_{X,k/N} \leqq \|p - p_J\|_{X,k/N} + \|p_J - \psi_N p_J\|_{X,k/N} + \|\psi_N p_J - \psi_N p\|_{X,k/N}$$
$$< \varepsilon/3 + J/N + \varepsilon/3 < \varepsilon \quad \text{providing } N > 3J/\varepsilon.$$

LEMMA 2. *If* $\|p - \psi_N p\|_{X,1/N} < \eta$, *then* $(\forall k \in \mathbf{N})$ $\|p - \psi_N p\|_{X,k/N} < \eta$.

PROOF.

$$\|p - \psi_N p\|_{X,k/N} = \frac{N}{k}\int_0^{k/N}\int_X |p(x,t) - \psi_N p(x,t)|\,d\mu dt$$
$$= \frac{1}{k}\sum_{i=1}^{k} N\int_0^{1/N}\int_X |p(x,t) - \psi_n p(x,t)|\,d\mu dt < \eta.$$

LEMMA 3. *If* $(\bar{S}, \bar{P} \vee \bar{Q})$ *and* $(S, P \vee Q)$ *are continuous processes and N is an integer such that*

(1) $(\bar{Q})_{\bar{S}} = (\bar{Q})_{\bar{S}_1}$, $(Q)_S = (Q)_{S_1}$ *and* $(\bar{S}, \bar{Q}) \sim (S, Q)$,

(2) $\|\bar{p} - \psi_N \bar{p}\|_{\bar{X},1/N} < \eta/3$ *and* $\|p - \psi_N p\|_{X,1/N} < \eta/3$, *and*

(3) $\bar{d}_{\bar{Q},Q}[(\bar{S}_{1/N}, \bar{P} \vee \bar{Q}), (S_{1/N}, P \vee Q)] < \eta/3$,

then $\bar{d}_{\bar{Q},Q}[(\bar{S}, \bar{P} \vee \bar{Q}), (S, P \vee Q)] < \eta$.

PROOF. Condition (3) implies that $(\forall k \in \mathbf{N})$ $\exists \phi : \bar{X} \to X$ which maps $(\bar{S}_{1/N}, \bar{Q}) \to (S_{1/N}, Q)$ isomorphically and with $\|\psi_N \bar{p}\phi^{-1} - \psi_N p\|_{X,k/N} < \eta/3$. Then, since ϕ preserves the flow-invariant factors generated by $\bar{Q}$ and Q we have

$$\bar{d}^{k/N}_{\bar{Q},Q}[(\bar{S}, \bar{P} \vee \bar{Q}), (S, P \vee Q)]$$
$$\leqq \|\bar{p}\phi^{-1} - p\|_{X,k/N}$$
$$\leqq \|\bar{p}\phi^{-1} - \psi_N \bar{p}\phi^{-1}\|_{X,k/N} + \|\psi_N \bar{p}\phi^{-1} - \psi_N p\|_{X,k/N} + \|\psi_N p - p\|_{X,k/N}$$
$$< \eta,$$

employing Lemma 2. Hence $\bar{d}_{\bar{Q},Q}[(\bar{S}, \bar{P} \vee \bar{Q}), (S, P \vee Q)] < \eta$.

The following theorem is the central copying theorem needed in the proof of the isomorphism theorem. It is the analogue in the present context of the "strong Sinai theorem" proved in [4]. Although it is a statement about continuous processes, a slight weakening of it has a natural restatement in terms only of flows and factors. This will be given following the proof.

THEOREM 1. *Let* $\bar{S}$ *be a flow on* $(\bar{X}, \bar{\mathcal{F}}, \bar{\mu})$ *with* $\bar{S}_1$ *ergodic. Let* $\bar{P}$ *and* $\bar{Q}$ *be partitions of* $\bar{X}$ *such that* $(\bar{S}_1, (\bar{Q})_{\bar{S}})$ *has a Bernoulli complement in* $(\bar{S}_1, (\bar{P} \vee \bar{Q})_{\bar{S}})$.

Let S *be a flow on* $(X, \mathcal{F}, \mu)$ *with* S_1 *ergodic and* $h(S, (P \vee Q)_S) = h(\bar{S}, (\bar{P} \vee \bar{Q})_{\bar{S}})$. *Let* P *and* Q *be partitions of* X *such that* $(S, Q) \sim (\bar{S}, \bar{Q})$ *and* $\bar{d}_{\bar{Q},Q}[(\bar{S}, \bar{P} \vee \bar{Q}), (S, P \vee Q)] < (\varepsilon/8)^2$.

Then $(\forall \xi > 0)$ *there exists a partition* $\tilde{P} \subset (P \vee Q)_S$ *such that* $|P - \tilde{P}| < \varepsilon$ *and* $\bar{d}_{\bar{Q},Q}[(\bar{S}, \bar{P} \vee \bar{Q}), (S, \tilde{P} \vee Q)] < \xi$.

PROOF. Without loss of generality we may assume that $(\bar{Q})_{\bar{S}_1} = (\bar{Q})_{\bar{S}}$, $(Q)_{S_1} = (Q)_S$ by replacing the given $\bar{Q}$ by a partition $\bar{Q}'$ that does generate under $\bar{S}_1$ and replacing Q by the image Q' of $\bar{Q}'$ under the isomorphism of $(\bar{S}, (\bar{Q})_{\bar{S}})$ and $(S, (Q)_S)$ that carries $\bar{Q}$ to Q. We simply note that this doesn't change the relative $\bar{d}$ distance between the processes.

Choose $\bar{\bar{P}} \supset \bar{P}$ so that $(\bar{\bar{P}} \vee \bar{Q})_{\bar{S}_1} = (\bar{\bar{P}} \vee \bar{Q})_{\bar{S}} = (\bar{P} \vee \bar{Q})_{\bar{S}}$. Fix $\xi > 0$. We may assume that $\xi < \varepsilon/100$. Choose $K \in \mathbf{N}$, so that $(\forall n \in \mathbf{N})$ $(\forall K' \geqq K)$ $\|\bar{\bar{p}} - \psi_{K'}\bar{\bar{p}}\|_{\bar{X},n} < (\xi/100)^2$. Choose $N \in \mathbf{N}$ a multiple of K, so that $K/N < \xi/100$. The discrete process $(\bar{S}_{1/N}, \bar{\bar{P}} \vee \bar{Q})$ is $\bar{Q}$-rel.f.d. so we may choose $\delta > 0$ and $u \in \mathbf{N}$, as in the definition of rel.f.d. for a tolerance of $\xi/3$. Then by Lemma 3, it will suffice to produce a partition $\tilde{\bar{\bar{P}}}$ of X such that

(i) $|\mathrm{dist} \bigvee_0^u \bar{S}_{-i/N}(\bar{\bar{P}} \vee \bar{Q}) - \mathrm{dist} \bigvee_0^u S_{-i/N}(\tilde{\bar{\bar{P}}} \vee Q)| < \delta$,

(ii) $|h(\bar{S}_{1/N}, \bar{\bar{P}} \vee \bar{Q}) - h(S_{1/N}, \tilde{\bar{\bar{P}}} \vee Q)| < \delta$,

(iii) $\|\tilde{\bar{\bar{p}}} - \psi_N \tilde{\bar{\bar{p}}}\|_{X,1/N} < \xi/3$,

(iv) $|P - \tilde{P}| < \varepsilon$ where $\tilde{P}$ is coded from $\tilde{\bar{\bar{P}}}$ exactly as $\bar{P}$ is from $\bar{\bar{P}}$.

Choose a partition $R \subset (P \vee Q)_S$ refining $P \vee Q$, such that

$$0 \leqq (h(\bar{S}_{1/N}, \bar{\bar{P}} \vee \bar{Q}) - h(S_{1/N}, R)) < \delta/10.$$

Choose $\rho > 0$ so that $\rho < \delta/100$, $\rho < \xi/100$ and

(1) for partitions R_1, R_2 of X labeled like R,

$$|R_1 - R_2| < \rho \Rightarrow |h(S_{1/N}, R_1) - h(S_{1/N}, R_2)| < \delta/10;$$

(2) for partitions P_1 and P_2 of X labeled like $P \vee Q$,

$$|P_1 - P_2| < \rho \Rightarrow |h(S_{1/N}, P_1) - h(S_{1/N}, P_2)| < \delta/10$$

and

$$\left| \mathrm{dist} \bigvee_0^u S_{-i/N} P_1 - \mathrm{dist} \bigvee_0^u S_{-i/N} P_2 \right| < \delta/10.$$

Now choose M, a multiple of N so that

$$4/M < \rho/100 \text{ and } (\forall n \in \mathbf{N}),$$

$$\|\bar{\bar{p}} - \psi_M \bar{\bar{p}}\|_{\bar{X},n} < \left(\frac{\xi}{100}\right)^2, \qquad \|p - \psi_M p\|_{X,n} < \left(\frac{\rho}{100}\right)^2,$$

$$\|q - \psi_M q\|_{X,n} < \left(\frac{\rho}{100}\right)^2 \quad \text{and} \quad \|r - \psi_M r\|_{X,n} < \left(\frac{\rho}{100}\right)^2.$$

Let

$$(3)\qquad \bar{\mathscr{A}} = \left\{ \bar{A} \in \bigvee_0^{L-1} \bar{S}_{-i/N}(\bar{\bar{P}} \vee \bar{Q}) : \sum \left| \frac{\#(\bar{A},\bar{C})}{L} - \bar{\mu}(\bar{C}) \right| < \frac{\delta}{10} \right\}$$

where the sum is taken over all $\bar{C} \in \bigvee_0^u \bar{S}_{-i/N}(\bar{\bar{P}} \vee \bar{Q})$ and where $\#(\bar{A},\bar{C})$ denotes the number of $i \in \{0,\cdots,L-1\}$ such that $(\forall \bar{x} \in \bar{A})\bar{S}_{i/N}\bar{x} \in \bar{C}$.

Then by the ergodic theorem and the Shannon–MacMillan theorem, we may choose L, a multiple of N, so that for a collection $\bar{\mathscr{B}} \subset \bigvee_0^{LM/N} \bar{S}_{i/M}(\bar{\bar{P}} \vee \bar{Q})$ with $\bar{\mu}(\bigcup \bar{\mathscr{B}}) > 1 - \rho/10$ we have

(4) $(\forall \bar{B} \in \bar{\mathscr{B}})\bar{S}_{i/M}\bar{B} \subset \bigcup \bar{\mathscr{A}}$, for $i = 0,1,2,\cdots,M/N-1$, and

(5) $\bar{\mu}(\bar{B}) < 2^{-(LM/N)(h(\bar{S}_{1/N},\bar{\bar{P}} \vee \bar{Q}) - \rho N/M)}$.

We also choose L so that for a collection of atoms $\mathscr{B} \subset \bigvee_0^{LM/N} S_{-i/M}(R)$ with $\mu(\bigcup \mathscr{B}) > 1 - \rho/10$ we have

$$(6)\qquad (\forall B \in \mathscr{B}) \quad 2^{-(LM/N)(h(S_{1/M},R)+\rho N/M)} < \mu(B).$$

Since

$$\begin{aligned} \|\psi_M\bar{\bar{p}} - \psi_K(\psi_M\bar{\bar{p}})\|_{\bar{X},L/N} &= \|\psi_M\bar{\bar{p}} - \psi_K\bar{\bar{p}}\|_{\bar{X},L/N} \\ &\leqq \|\psi_M\bar{\bar{p}} - \bar{\bar{p}}\|_{\bar{X},L/N} + \|\bar{\bar{p}} - \psi_K\bar{\bar{p}}\|_{\bar{X},L/N} \\ &< (\xi/100)^2 + (\xi/100)^2 < (\xi/50)^2, \end{aligned}$$

we have

$$\bar{\mu}\{\bar{x} \in \bar{X} \mid \|\psi_M\bar{\bar{p}}(\bar{x}) - \psi_K(\psi_M\bar{\bar{p}})(\bar{x})\|_{L/N} < (\xi/50)\} > 1 - (\xi/50).$$

Hence if $\bar{\mathscr{B}}_0 = \{\bar{B} \in \bar{\mathscr{B}} \mid \|\psi_M\bar{\bar{p}}(\bar{B}) - \psi_K\bar{\bar{p}}(\bar{B})\|_{L/N} < (\xi/50)\}$ then $\bar{\mu}(\bigcup \bar{\mathscr{B}}_0) > 1 - \xi/50 - \rho/10$.

Let $\varphi : \bar{X} \to X$ provide a relative $\bar{d}$ match between $(\bar{S}_{1/M}, \bar{P} \vee \bar{Q})$ and $(S_{1/M}, P \vee Q)$ to $(\varepsilon/8)^2$ along $[0, 1/M, \cdots, L/N]$. Then if

$$H_1 = \{x \in X \mid \|\psi_M\bar{p}(\varphi^{-1}(x)) - \psi_M p(x)\|_{L/N} < \varepsilon/8\}$$

we have $\mu(H_1) > 1 - \varepsilon/8$. If $H_2 = \varphi(\bigcup \bar{\mathscr{B}}_0)$, then $\mu(H_2) > 1 - \xi/50 - \rho/10$. Let $\mathscr{B}_0 = \{B \in \mathscr{B} \mid \mu(B \cap H_1 \cap H_2) > \frac{1}{2}\mu(B)\}$ so that $\mu(\bigcup \mathscr{B}_0) > 1 - \varepsilon/4 + \xi/25 - 3\rho/10$.

The Shannon–MacMillan conditions (5) and (6) tell us that $(\forall \bar{B} \in \bar{\mathscr{B}})$ $(\forall B \in \mathscr{B})$,

$$\frac{\mu(\bar{B})}{\mu(B)} < 2^{-L[h(\bar{S}_{1/N},\bar{\bar{P}} \vee \bar{Q}) - (M/N)h(S_{1/M},R) - 2\rho]} < 2^{2L\rho}$$

so if we divide each $\bar{B} \in \bar{\mathscr{B}}$ into $2^{(2L\rho+1)}$ pieces $\{\bar{B}_i\}_{i=1}^{2^{(2L\rho+1)}}$ of equal measure, we may apply the marriage lemma [6] to establish a 1-1 map $\lambda : \mathscr{B}_0 \to \{$pieces of atoms of $\bar{\mathscr{B}}_0\}$ such that $(\forall B \in \mathscr{B}_0)$ $\mu(B \cap H_1 \cap \varphi(\lambda(B))) > 0$.

Now let $\mathscr{C}_1 = \{C \in \bigvee_0^{LM/N} S_{-i/M}(Q) \mid \mu(C \cap H_2) > \frac{1}{2}\mu(C)\}$ so that $\mu(\bigcup \mathscr{C}_1) > 1 - \xi/25 - 2\rho/10$. Let $\mathscr{B}_1' = \{B \in \mathscr{B} \mid (\exists C \in \mathscr{C}_1),\ B \subset C\}$ and $\mathscr{B}_1 = \mathscr{B}_0 \cup \mathscr{B}_1'$ so that $\mu(\bigcup \mathscr{B}_1) > 1 - \xi/25 - 3\rho/10$.

The conditions (5) and (6) tell us that $(\forall C \in \mathscr{C}_1)$, there are more φ-images of pieces of $\bar{\mathscr{B}}_0$-atoms in C than $\mathscr{B}_1$-atoms in C, and since λ respects $\bar{Q} - Q - LM/N$ names, we may extend λ to $\mathscr{B}_1$ in a 1-1 fashion so that $\lambda : \mathscr{B}_1 \to$ {pieces of $\bar{\mathscr{B}}_0$ atoms} and λ still respects $\bar{Q} - Q - LM/N$ names.

Finally, let $\mathscr{C}_2 = \{C \in \bigvee_0^{LM/N} S_{-i/M}(Q) \mid \mu(C \cap \varphi(\bigcup \bar{\mathscr{B}})) > \frac{1}{2}\mu(C)\}$ so that $\mu(\bigcup \mathscr{C}_2) > 1 - 2\rho/10$ and let $\mathscr{B}_2' = \{B \in \mathscr{B} \mid (\exists C \in \mathscr{C}_2) B \subset C\}$ and $\mathscr{B}_2 = \mathscr{B}_1 \cup \mathscr{B}_2'$ so that $\mu(\bigcup \mathscr{B}_2) > 1 - 3\rho/10$.

Again we may extend λ so that $\lambda : \mathscr{B}_2 \to$ {pieces of $\bar{\mathscr{B}}$ atoms} in a 1-1 fashion while still preserving $\bar{Q} - Q - LM/N$ names.

Now let $H_3 = \{x \mid \|p(x) - \psi_M p(x)\|_{L/N} < \rho/100$, and $\|q(x) - \psi_M q(x)\|_{L/N} < \rho/100$, and $\|r(x) - \psi_M r(x)\|_{L/N} < \rho/100\}$ so that $\mu(H_3) > 1 - 3\rho/100$.

To construct $\bar{\bar{P}}$ we build a Rokhlin tower (T, F, μ_F) in $(S, X, (P \vee Q)_S, \mu)$ of height L/N such that $\mu(T) > 1 - \rho/100$, and such that

$$\left| \operatorname*{dist}_{F,\mu_F}\left(\bigvee_0^{LM/N} S_{-i/M}(R) \vee \{H_3, H_3'\}\right) - \operatorname*{dist}_X\left(\bigvee_0^{LM/N} S_{-i/M}(R) \vee \{H_3, H_3'\}\right)\right| < \frac{\rho}{100}.$$

Let $F_i = F \cap H_3 \cap (\bigcup B_i)$, $i = 0, 1, 2$. Then $F_0 \subset F_1 \subset F_2$, and we readily compute that $\mu_F(F_0) > (1 - \varepsilon/3)\mu_F(F)$, $\mu_F(F_1) > (1 - \xi/10)\mu_F(F)$, and $\mu_F(F_2) > (1 - 2\rho/5)\mu_F(F)$.

Let $T_i = \bigcup_{t=0}^{L/N} S_t F_i$, $i = 0, 1, 2$.

Before defining $\bar{\bar{P}}$, it will be useful to introduce three new partitions of X, $\hat{P}$, $\hat{Q}$, and $\hat{R}$. We define them on T_2 by putting $(\forall x \in F_2)$ $(\forall t \in [0, L/N])$.

$$\hat{p}(x,t) = \psi_M p(x,t),$$

$$\hat{q}(x,t) = \psi_M q(x,t)$$

and

$$\hat{r}(x,t) = \psi_M r(x,t).$$

We include the remainder of the space in the sets $\hat{P}_1$, $\hat{Q}_1$ and $\hat{R}_1$, respectively. Note that on T_2, these partitions make the tower look like one arising from the discrete process $(S_{1/M}, P \vee Q \vee R)$ as in Thouvenot [9]. Furthermore, we compute that $|\hat{P} - P| < \rho$, $|\hat{Q} - Q| < \rho$, and $|\hat{R} - R| < \rho$. Indeed, if $x \in F_2$, then $x \in H_3$, so that, for example, $\|\psi_M p(x) - p(x)\|_{L/N} < \rho/100$. Since $\mu(T_2) > 1 - 2\rho/5 - \rho/100$ we have that $|\hat{P} - P| < \rho$.

For $x \in F_2$, let $B(x)$ denote the atom of $\mathscr{B}$ containing x. We define $\bar{\bar{P}}$ on T_2 by putting $(\forall x \in F_2)$ $(\forall t \in [0, L/N])$ $\bar{\bar{p}}(x,t) = \psi_M \bar{p}(\lambda(B(x)))$, and we include the remainder of the space in one of the sets of $\bar{\bar{P}}$. We now verify that conditions (i) through (iv) hold.

(i): $(\forall x \in F_2)$ we see that the $S_{1/M} - \hat{Q}$-name of x along $[0, 1/M, 2/M, \cdots, L/M]$ coincides with the $\bar{S}_{1/M} - \bar{Q}$-name of $\lambda(B(x))$ along $[0, 1/M, 2/M, \cdots, L/N]$, so conditions (3) and (4) imply that

$$\left| \operatorname{dist}_{T_2} \left(\bigvee_0^u S_{-i/N}(\bar{\bar{P}} \vee \hat{Q}) \right) - \operatorname{dist}_{\bar{X}} \left(\bigvee_0^u \bar{S}_{-i/N}(\bar{\bar{P}} \vee \bar{Q}) \right) \right| < \delta/10$$

and since $\mu(T_2) > 1 - 2\rho/5 - \rho/100$

$$\left| \operatorname{dist}_{X} \left(\bigvee_0^u S_{-i/N}(\bar{\bar{P}} \vee \hat{Q}) \right) - \operatorname{dist}_{\bar{X}} \left(\bigvee_0^u \bar{S}_{-i/N}(\bar{\bar{P}} \vee \bar{Q}) \right) \right| < \delta/10 + 2\rho/5 + \rho/100. \tag{7}$$

But $|\hat{Q} - Q| < \rho \Rightarrow |\bar{\bar{P}} \vee \hat{Q} - \bar{\bar{P}} \vee Q| < \rho$, so by the choice of ρ (cf. (2)) we have

$$\left| \operatorname{dist}_{X} \left(\bigvee_0^u S_{-i/N}(\bar{\bar{P}} \vee Q) \right) - \operatorname{dist}_{X} \left(\bigvee_0^u S_{-i/N}(\bar{\bar{P}} \vee \hat{Q}) \right) \right| < \delta/10$$

and this with (7) gives (i).

(ii): We have $|\bar{\bar{P}} \vee Q - \bar{\bar{P}} \vee \hat{Q}| < \rho \Rightarrow h(S_{1/N}, \bar{\bar{P}} \vee Q) > h(S_{1/N}, \bar{\bar{P}} \vee \hat{Q}) - \delta/10$. Let W be the partition of X defined on $\bigcup_0^{1/N} S_t F_2$ by setting $(\forall x \in F_2$, $t \in [0, 1/N))$ $w(x,t) = (i,j)$ providing $t \in [j/M, (j+1)/M)$ and $\lambda(B(x)) = \bar{B}_i$, the ith piece (from among at most $2^{2L\rho+1}$) of an atom $\bar{B} \in \bar{\mathscr{B}}$, and by setting $w(y) = l$ (some fixed symbol) for $y \notin \bigvee_0^{1/N} S_t F_2$. A simple computation shows that for sufficiently large choice of L, $h(W) < \delta/10$, so that $h(S_{1/N}, \bar{\bar{P}} \vee \hat{Q}) > h(S_{1/N}, \bar{\bar{P}} \vee \hat{Q} \vee W) - \delta/10$. Now we see that $\bigvee_{-L}^{L} S_{i/N}(\bar{\bar{P}} \vee \hat{Q} \vee W)$ refines $\hat{R}$ on T_2 and since $\mu(T_2) > 1 - 2\rho/5 - \rho/100$, we have $\bigvee_{-L}^{L} S_{i/N}(\bar{\bar{P}} \vee \hat{Q} \vee W) \supset_\rho \hat{R}$ so that by the choice of ρ (cf. (1)) we have $h(S_{1/N}, \bar{\bar{P}} \vee \hat{Q} \vee W) > h(S_{1/N}, \hat{R}) - \delta/10$. Further, $|\hat{R} - R| < \rho$ so that $h(S_{1/N}, \hat{R}) > h(S_{1/N}, R) - \delta/10$. But $h(S_{1/N}, R) > h(\bar{S}_{1/N}, \bar{\bar{P}} \vee \bar{Q}) - \delta/10$, so that in summary, $h(S_{1/N}, \bar{\bar{P}} \vee Q) > h(\bar{S}_{1/N}, \bar{P} \vee \bar{Q}) - \delta$. Since $h(S_{1/N}, \bar{\bar{P}} \vee Q) \leqq h(\bar{S}_{1/N}, \bar{\bar{P}} \vee \bar{Q})$, this establishes (ii).

(iii): If $x \in F_1$, then $B(x) \in \mathscr{B}_1$ and $\lambda(B(x)) \in \bar{\mathscr{B}}_0$ so that

$$\|\bar{\bar{p}}(x) - \psi_K \bar{\bar{p}}(x)\|_{L/N} = \|\psi_M \bar{p}(\lambda(B(x))) - \psi_K \bar{p}(\lambda(B(x)))\|_{L/N} < \xi/50.$$

That is,

$$\begin{aligned} &\frac{N}{L} \int_0^{L/N} |\bar{\bar{p}}(x,t) - \psi_K \bar{\bar{p}}(x,t)|\, dt \\ &= \frac{N}{LK} \sum_{i=0}^{LK/N-1} \int_{i/K}^{(i+1)/K} |\bar{\bar{p}}(x,t) - \bar{\bar{p}}(x, i/K)|\, dt < \xi/50. \end{aligned} \tag{8}$$

We have $(\forall x \in F_1)$

$$\frac{N}{L}\int_0^{L/N} N\int_0^{1/N} |\bar{\bar{p}}(S_t x, s) - \psi_N \bar{\bar{p}}(S_t x, s)|\, ds dt$$

$$= N\int_0^{1/N} \frac{N}{L}\int_0^{L/N} |\bar{\bar{p}}(x, s+t) - \bar{\bar{p}}(x,t)|\, dt ds$$

$$\leqq N\int_0^{1/N} \frac{N}{L}\int_0^{L/N} |\bar{\bar{p}}(x, s+t) - \psi_K \bar{\bar{p}}(x,t)|\, dt ds$$

$$(9) \qquad + N\int_0^{1/N} \frac{N}{L}\int_0^{L/N} |\psi_K \bar{\bar{p}}(x,t) - \bar{\bar{p}}(x,t)|\, dt ds$$

$$\leqq N\int_0^{1/N} \frac{N}{L}\int_0^{L/N} |\bar{\bar{p}}(x, s+t) - \psi_K \bar{\bar{p}}(x,t)|\, dt ds + \xi/50$$

$$= N\int_0^{1/N} \frac{N}{LK}\left[\sum_{i=0}^{LK/N-1} K\int_{i/K}^{(i+1)/K} |\bar{\bar{p}}(x, s+t) - \bar{\bar{p}}(x, i/K)|\, dt\right] ds + \xi/50.$$

But $(\forall s \in [0, 1/N))$,

$$\frac{N}{LK}\sum_{i=0}^{LK/N-1} K\int_{i/K}^{(i+1)/K} |\bar{\bar{p}}(x, s+t) - \bar{\bar{p}}(x, i/K)|\, dt$$

$$\leqq \frac{N}{LK}\sum_{i=0}^{LK/N-1} \left[K\int_{i/K}^{(i+1)/K-s} |\bar{\bar{p}}(x, s+t) - \bar{\bar{p}}(x, i/K)|\, dt + \frac{2K}{N}\right]$$

which, because of (8) is $\leqq \xi/50 + 2K/N < \xi/50 + \xi/50$ so that the quantity (9) is $< \xi/50 + \xi/50$. Therefore,

$$\|\bar{\bar{p}} - \psi_N \bar{\bar{p}}\|_{T_1, 1/N} < \xi/50 + \xi/50$$

and since $\mu(T_1) > 1 - \xi/10 - \rho/100$, $\|\bar{\bar{p}} - \psi_N \bar{\bar{p}}\|_{X, 1/N} < \xi/50 + \xi/50 + \xi/5 + \rho/50 < \xi/3$.

(iv): Let $\tilde{P}$ be coded from $\bar{\bar{P}}$ exactly as $\bar{P}$ is coded from $\bar{\bar{P}}$. Then $(\forall x \in F_0)$ $\|\tilde{p}(x) - p(x)\|_{L/N} = \|\psi_M \bar{p}(\lambda(B(x))) - \psi_M p(x)\|_{L/N} < \varepsilon/8$ since $B(x) \in \mathcal{B}_0$, and λ was defined on $\mathcal{B}_0$ to give a good $\bar{d}$-match. But $\mu(T_0) > 1 - \varepsilon/3 - \rho/100$ so $|\tilde{P} - \hat{P}| < \varepsilon/8 + 2\varepsilon/3 + \rho/50$. Since $|\hat{P} - P| < \rho$ we have $|\tilde{P} - P| < \varepsilon/8 + 2\varepsilon/3 + \rho/50 + \rho < \varepsilon$.

This concludes the proof. The following consequence of this theorem is the result we will actually use in proving the relative isomorphism theorem.

COROLLARY 1. *Let* $(\bar{S}, \bar{P} \vee \bar{Q})$ *and* $(S, P \vee Q)$ *be as in Theorem* 1. *Then there* *exists a partition $\tilde{P}$ of X such that $\bar{d}_{\bar{Q},Q}[(\bar{S},\bar{P}\vee\bar{Q}),(S,P\vee Q)]=0$, and $|P-\tilde{P}|<2\varepsilon$.*

PROOF. Simply apply Theorem 1 repeatedly to obtain a sequence of partitions $\tilde{P}_i$, $i=1,2,3,\cdots$ such that $|P-\tilde{P}_1|+\Sigma_{i=1}^{\infty}|\tilde{P}_i-\tilde{P}_{i+1}|<2\varepsilon$,

$$\bar{d}_{\bar{Q},Q}[(\bar{S},\bar{P}\vee\bar{Q}),(S,\tilde{P}_i\vee Q)]\to 0 \qquad \text{as } i\to\infty.$$

Then $\tilde{P}=\lim_{i\to\infty}\tilde{P}_i$ will serve as the desired partition.

REMARK 1. We observe now that we have obtained a result of sufficient interest to merit its independent statement in somewhat more natural terms.

THEOREM 1'. *Let $\bar{S}$ be a flow on $(\bar{X},\bar{\mathcal{F}},\bar{\mu})$ with $\bar{S}_1$ ergodic. Let $\bar{\mathcal{C}}\subset\bar{\mathcal{F}}$ be a factor of $\bar{S}$ such that $(\bar{S}_1,\bar{\mathcal{C}})$ has a Bernoulli complement in $(\bar{S}_1,\bar{\mathcal{F}})$. Let S be a flow on $(X,\mathcal{F},\mu)$ with S_1 ergodic and $h(S_1,\mathcal{F})=h(\bar{S}_1,\bar{\mathcal{F}})$. Let $\mathcal{C}\subset\mathcal{F}$ be a factor of S such that the factor actions $(S,\mathcal{C})$ and $(\bar{S},\bar{\mathcal{C}})$ are isomorphic.*

Then given an isomorphism $\phi:(\bar{S},\bar{\mathcal{C}})\to(S,\mathcal{C})$ there is a factor $\mathcal{F}'\subset\mathcal{F}$ of S such that $(S,\mathcal{F}')$ and $(\bar{S},\bar{\mathcal{F}})$ are isomorphic via an isomorphism that extends ϕ.

PROOF. Let $\bar{Q}$ generate $\bar{\mathcal{C}}$ under $\bar{S}$ and set $Q=\phi(\bar{Q})$. Choose partitions $\bar{P}$ of $\bar{X}$ and P of X so that $(\bar{P}\vee\bar{Q})_{\bar{S}}=\bar{\mathcal{F}}$ and $(P\vee Q)_S=\mathcal{F}$. We apply Corollary 1 to obtain a partition $\tilde{P}$ so that $\bar{d}_{\bar{Q},Q}[(\bar{S},\bar{P}\vee\bar{Q}),(S,\tilde{P}\vee Q)]=0$. Then $(\tilde{P}\vee Q)_S$ is the desired factor.

REMARK 2. In fact we can prove Theorem 1' in the case where $h(S_1,\mathcal{F})>h(\bar{S}_1,\bar{\mathcal{F}})$. In this case, we consider the direct product of $(\bar{S},\bar{\mathcal{C}})$ with a Bernoulli flow $(\bar{S},\bar{\mathcal{B}}')$ such that $h(S_1',\bar{\mathcal{B}}')+h(\bar{S}_1,\bar{\mathcal{C}})=h(S_1,\mathcal{F})$. We apply Theorem 1' to this direct product to obtain a Bernoulli factor $(S,\mathcal{B}')$ of $(S,\mathcal{F})$ independent of $\mathcal{C}$. Now we take a factor B'' of $(S,\mathcal{B}')$ so that $h(S_1,\mathcal{B}'')+h(S_1,\mathcal{C})=h(S_1,\mathcal{B}''\vee\mathcal{C})=h(\bar{S}_1,\bar{\mathcal{F}})$. Finally, we apply Theorem 1' to $(\bar{S},\bar{\mathcal{F}})$ and $(S,\mathcal{B}''\vee\mathcal{C})$ to get the result.

REMARK 3. We can also obtain an extension of the approximation result Corollary 1 to the case when $h(S,(P\vee Q)_S)>h(\bar{S})$. In this case let $(S'(B')_{S'})$ be a Bernoulli flow with a two set (flow) generator B' such that $h(S')=h(S,(P\vee Q)_S)-h(\bar{S})$. Form the direct product $\hat{S}=S'\times\bar{S}$ with partitions $\hat{B}$, $\hat{P}$ and $\hat{Q}$ corresponding to B', $\bar{P}$ and $\bar{Q}$. We may choose the distribution of B' so that we still have

$$\bar{d}_{\hat{Q},Q}[(\hat{S},\hat{B}\vee\hat{P}\vee\hat{Q}),(S,0\vee P\vee Q)]<(\varepsilon/8)^2$$

where 0 is the trivial partition on X, indexed like $\hat{B}$. Now apply Corollary 1 to obtain partitions $\tilde{B}$ and $\tilde{P}$ of X satisfying $(\tilde{S}, \tilde{B} \vee \hat{P} \vee \hat{Q}) \sim (S, \tilde{B} \vee \tilde{P} \vee Q)$ and $|0 \vee P - \tilde{B} \vee \tilde{P}| < \varepsilon$. Then $\tilde{P}$ is the desired partition.

The remainder of the proof of the relative isomorphism theorem is carried out somewhat in the spirit of A. Rothstein's presentation of the isomorphism theorem for Bernoulli shifts [7]. The main step is the following Lemma, which describes a modification of joint continuous processes that preserves the joint distributions of the associated discrete process indexed by the integers.

LEMMA 4. *Let $(S, P \vee Q \vee \tilde{P})$ be a process $(X, \mathscr{F}, \mu)$ such that $(P \vee Q)_S = \mathscr{F}$, S_1 is ergodic, $(Q)_S$ has a Bernoulli complement in $(S_1, \mathscr{F})$ and $h(S_1) = h(S_1, (\tilde{P} \vee Q)_S)$.*

Suppose that π and $\tilde{\pi}$ are codes of $\bigvee_{-N_0}^{N_0} S_i(P \vee Q)$ and $\bigvee_{-\tilde{N}_0}^{\tilde{N}_0} S_i(\tilde{P} \vee Q)$ respectively such that $|\pi(P \vee Q) - \tilde{P}| < \varepsilon$ and $|\tilde{\pi}(\tilde{P} \vee Q) - P| < \tilde{\varepsilon}$. Then there is a partition $\mathbf{P} \subset (\tilde{P} \vee Q)_S$ such that $\bar{d}_{Q,Q}[(S, \mathbf{P} \vee Q), (S, P \vee Q)] = 0$ and $|\pi(\mathbf{P} \vee Q) - \tilde{P}| < 7\varepsilon$ and $|\tilde{\pi}(\tilde{P} \vee Q) - \mathbf{P}| < 7\tilde{\varepsilon}$.

PROOF. As in the proof of Theorem 1, we may assume, without loss of generality, that $(Q)_{S_1} = (Q)_S$.

Choose partitions $P^* \supset P$ and $\tilde{P}^* \supset \tilde{P}$ so that $(P^* \vee Q)_{S_1} = (P^* \vee Q)_S$ and $(\tilde{P}^* \vee Q)_{S_1} = (\tilde{P}^* \vee Q)_S = (\tilde{P} \vee Q)_S$. Let λ and $\tilde{\lambda}$ be the codes on P^* and $\tilde{P}^*$ such that $\lambda(P^*) = P$ and $\tilde{\lambda}(\tilde{P}^*) = \tilde{P}$.

Choose $\eta > 0$, $\eta < \tilde{\varepsilon}$ so that if two partitions R_1 and R_2 satisfy $|R_1 - R_2| < \eta$ then $|\pi\lambda(R_1) - \pi\lambda(R_2)| < \varepsilon$ and $|\tilde{\pi}\tilde{\lambda}(R_1) - \tilde{\pi}\tilde{\lambda}(R_2)| < \tilde{\varepsilon}$. Then by the Corollary to Theorem 1, it will suffice to produce a partition $\mathbf{P}^* \subset (\tilde{P} \vee Q)_S$ such that

(i) $\bar{d}_{Q,Q}[(S, \mathbf{P}^* \vee Q), (S, P^* \vee Q)] < \xi = (\eta/10)^2$ and

(ii) $|\pi\lambda(\mathbf{P}^* \vee Q) - \tilde{P}| < 6\varepsilon$ and $|\tilde{\pi}\tilde{\lambda}(\tilde{P}^* \vee Q) - \lambda(\mathbf{P}^*)| < 6\tilde{\varepsilon}$

for if we apply Corollary 1 and obtain a partition $\mathbf{P}^{**}$ such that $|\mathbf{P}^{**} - \mathbf{P}^*| < \eta$ and $\bar{d}_{Q,Q}[(S, \mathbf{P}^{**} \vee Q), (S, P^* \vee Q)] = 0$ then $\lambda(\mathbf{P}^{**})$ will serve as the desired partition $\mathbf{P}$.

Choose $N \in \mathbf{N}$ so that $(\forall n)\ \|p^* - \psi_N p^*\|_{X, n/N} < \xi/100$.

Choose $\delta > 0$ and $u \in \mathbf{N}$ for $(S_{1/N}, P^* \vee Q)$ as in the definition of rel.f.d. for a tolerance of $\xi/3$. Then by Lemma 3, it will suffice to produce a partition $\mathbf{P}^*$ in $(\tilde{P} \vee Q)_S$ such that

(iii) $|\pi\lambda(\mathbf{P}^* \vee Q) - \tilde{P}| < 6\varepsilon$, $|\tilde{\pi}\tilde{\lambda}(\tilde{P}^* \vee Q) - \lambda(\mathbf{P}^*)| < 6\tilde{\varepsilon}$,

(iv) $|\mathrm{dist}_X \bigvee_0^{u-1} S_{-i/N}(\mathbf{P}^* \vee Q) - \mathrm{dist}_X \bigvee_0^{u-1} S_{-i/N}(P^* \vee Q)| < \delta$,

(v) $|h(S_{1/N}, \mathbf{P}^* \vee Q) - h(S_{1/N}, P^* \vee Q)| < \delta$ and

(vi) $\|\mathbf{p}^* - \psi_N \mathbf{p}^*\|_{X, 1/N} < \xi/3$.

Choose $\rho_0 > 0$ so that for all partitions R_1 and R_2 of X, $|R_1 - R_2| < \rho_0$ implies $|h(S_{1/N}, R_1) - h(S_{1/N}, R_2)| < \delta$.

Choose $N_1 \in \mathbf{N}$ so that for some code π_1 on $\bigvee_{-N_1}^{N_1} S_i(P^* \vee Q)$ we have $|\pi_1(P^* \vee Q) - \tilde{P}^*| < \rho_0/100$.

Choose $\rho_1 > 0$, $\rho_1 < \min(\varepsilon, \tilde{\varepsilon}, \xi/200, \delta, \rho_0)$ so that for all partitions R_1 and R_2 of X, $|R_1 - R_2| < \rho_1$ implies

$$|\pi_1(R_1) - \pi_1(R_2)| < \frac{\rho_0}{100}, \quad |\pi\lambda(R_1) - \pi\lambda(R_2)| < \varepsilon, \quad |\tilde{\pi}\tilde{\lambda}(R_1) - \tilde{\pi}\tilde{\lambda}(R_2)| < \tilde{\varepsilon},$$

and

$$\left| \operatorname*{dist}_X \bigvee_0^{u-1} S_{-i/N}(R_1) - \operatorname*{dist}_X \bigvee_0^{u-1} S_{-i/N}(R_2) \right| < \frac{\delta}{100}.$$

Now choose $M \in \mathbf{N}$ so that

$$\|p^* - \psi_M p^*\|_{x,1/M} < \left(\frac{\rho_1}{200}\right)^2, \quad \|\tilde{p}^* - \psi_M \tilde{p}^*\|_{x,1/M} < \left(\frac{\rho_1}{200}\right)^2, \quad \text{and}$$

$$\|q - \psi_M q\|_{x,1/M} < \left(\frac{\rho_1}{200}\right)^2$$

and let $H_1 = \{x \in X \mid \|p^* - \psi_M p^*\|_{x,1/M} < \rho_1/200,\ \|\tilde{p}^* - \psi_M \tilde{p}^*\|_{x,1/M} < \rho_1/200$ and $\|q - \psi_M q\|_{x,1/M} < \rho_1/200\}$ and $H_2 = \{x \in X \mid \|\tilde{p}^* - \psi_M \tilde{p}^*\|_{x,1/M} < \rho_2/200$ and $\|q - \psi_M q\|_{x,1/M} < \rho_1/200\}$, so that $\mu(H_1) > 1 - 3\rho_1/200$ and $\mu(H_2) > 1 - 2\rho_1/200$. Now choose $L \in \mathbf{N}$ so that $(N + N_1 + u/N)/L < \rho_1/200$ and construct the following Rokhlin towers: (T, F, μ_F) in $(X, \mathcal{F}, \mu)$ and $(\tilde{T}, \tilde{F}, \mu_{\tilde{F}})$ in $(X, (\tilde{P} \vee Q)_S, \mu)$, both of height L and so that $\mu(T) > 1 - \rho_1/20$, $\mu(\tilde{T}) > 1 - \rho_1/20$, $F \subset H_1$ and $\tilde{F} \subset H_2$ and

(vii) $\operatorname{dist}_F \bigvee_0^{LM-1} S_{-i/M}(\tilde{P}^* \vee Q) = \operatorname{dist}_{\tilde{F}} \bigvee_0^{LM-1} S_{-i/M}(\tilde{P}^* \vee Q)$.

Using the tower T we construct partitions $\boldsymbol{P}^*$, $\boldsymbol{Q}$ and $\tilde{\boldsymbol{P}}^*$ by setting $(\forall x \in F)$ and $(\forall t \in [0, L])$ $\boldsymbol{p}^*(x, t) = \psi_M p^*(x, t)$, $\boldsymbol{q}(x, t) = \psi_M q(x, t)$, and $\tilde{\boldsymbol{p}}^*(x, t) = \psi_M \tilde{p}^*(x, t)$, and by including $X \setminus T$ in the first sets of $\boldsymbol{P}^*$, $\boldsymbol{Q}$, and $\tilde{\boldsymbol{P}}^*$. We see then that $|\boldsymbol{P}^* - P^*| < 11\rho_1/100$, $|\boldsymbol{Q} - Q| < 11\rho_1/100$ and $|\tilde{\boldsymbol{P}}^* - \tilde{P}^*| < 11\rho_1/100$. Indeed, $(\forall x \in F)$ we have, for example, $\|\boldsymbol{p}^*(x, t) - p^*(x, t)\|_{x,L} < p_1/100$, and since $\mu(T) > 1 - \rho_1/100$, $\|\boldsymbol{p}^*(x) - p^*(x)\|_X < 11\rho_1/100$.

In the same way, using $\tilde{T}$ we construct partitions $\tilde{\mathbf{P}}^*$ and $\mathbf{Q}$ so that $|\tilde{\mathbf{P}}^* - \tilde{P}^*| < 11\rho_1/100$ and $|\mathbf{Q} - Q| < 11\rho_1/100$. Now because of (vii), we may take a bimeasurable map $\phi : \tilde{F} \to F$ preserving the normalized measures $\mu_{\tilde{F}}$ and μ_F such that $(\forall x \in \tilde{F})$ $(\forall t \in [0, L])$ $\tilde{\mathbf{p}}^* \vee \mathbf{q}(x, t) = \tilde{\boldsymbol{p}}^* \vee \boldsymbol{q}(\phi(x), t)$. We then define $\mathbf{P}^*$ by setting $(\forall x \in \tilde{F})$ $(\forall t \in [0, L])$ $\mathbf{p}^*(x, t) = \boldsymbol{p}(\phi(x), t)$, and by putting $X \setminus \tilde{T}$ in the first set of $\mathbf{P}^*$. We now verify that conditions (iii) through (vi) hold.

(iii): We have $|\pi\lambda(P^* \vee Q) - \tilde{P}| < \varepsilon$, but since $|\boldsymbol{P}^* \vee \boldsymbol{Q} - P^* \vee Q| < \rho_1$ and $|\tilde{\lambda}(\tilde{\boldsymbol{P}}^*) - \tilde{P}| < \rho_1 < \varepsilon$ we have $|\pi\lambda(\boldsymbol{P}^* \vee \boldsymbol{Q}) - \tilde{\lambda}(\tilde{\boldsymbol{P}}^*)| < 3\varepsilon$. Then by the construction of $\mathbf{P}^*$, $\mathbf{Q}$ and $\tilde{\mathbf{P}}^*$, and since $\mu(\bigvee_{N_0}^{L-N_0} S_t\tilde{F}) > 1 - 6\rho_1/100$ we have

$$|\pi\lambda(\mathbf{P}^* \vee \mathbf{Q}) - \tilde{\lambda}(\tilde{\mathbf{P}}^*)| < 3\varepsilon + 12\rho_1/100.$$

But $|\mathbf{P}^* \vee \mathbf{Q} - \mathbf{P}^* \vee Q| < \rho_1$ and $|\tilde{\lambda}(\tilde{\mathbf{P}}^*) - \tilde{P}| < \rho_1 < \varepsilon$ so $|\pi\lambda(\mathbf{P}^* \vee Q) - \tilde{P}| < 6\varepsilon$. In exactly the same way one sees that $|\tilde{\pi}\tilde{\lambda}(\tilde{P}^* \vee Q) - \lambda(\mathbf{P}^*)| < 6\tilde{\varepsilon}$.

(iv): We have $|P^* \vee Q - \boldsymbol{P}^* \vee \boldsymbol{Q}| < \rho_1$ so that

$$\left|\operatorname{dist} \bigvee_0^{u-1} S_{-i/N}(P^* \vee Q) - \operatorname{dist} \bigvee_0^{u-1} S_{-i/N}(\boldsymbol{P}^* \vee \boldsymbol{Q})\right| < \delta/100.$$

But by the construction of $\mathbf{P}^*$ and $\mathbf{Q}$ and since $\mu(\bigcup_{u/N}^{L-u/N} S_tF) > 1 - 6\rho_1/100 > 1 - 6\delta/100$ and $\mu(\bigcup_{u/N}^{L-u/N} S_t\tilde{F}) > 1 - 6\delta/100$ we have (denoting these portions of T and $\tilde{T}$ by E and $\tilde{E}$)

$$\left|\operatorname*{dist}_X \bigvee_0^{u-1} S_{-i/N}(\boldsymbol{P}^* \vee \boldsymbol{Q}) - \operatorname*{dist}_E \bigvee_0^{u-1} S_{-i/N}(\boldsymbol{P}^* \vee \boldsymbol{Q})\right| < 12\delta/100,$$

$$\left|\operatorname*{dist}_E \bigvee_0^{u-1} S_{-i/N}(\boldsymbol{P}^* \vee \boldsymbol{Q}) - \operatorname*{dist}_{\tilde{E}} \bigvee_0^{u-1} S_{-i/N}(\mathbf{P}^* \vee \mathbf{Q})\right| = 0,$$

$$\left|\operatorname*{dist}_{\tilde{E}} \bigvee_0^{u-1} S_{-i/N}(\mathbf{P}^* \vee \mathbf{Q}) - \operatorname*{dist}_X \bigvee_0^{u-1} S_{-i/N}(\mathbf{P}^* \vee \mathbf{Q})\right| < 12\delta/100$$

and since $|\mathbf{P}^* \vee \mathbf{Q} - \mathbf{P}^* \vee Q| < \rho_1$,

$$\left|\operatorname*{dist}_X \bigvee_0^{u-1} S_{-i/N}(\mathbf{P}^* \vee \mathbf{Q}) - \operatorname*{dist}_X \bigvee_0^{u-1} S_{-i/N}(\mathbf{P}^* \vee Q)\right| < \delta/100.$$

Therefore

$$\left|\operatorname*{dist}_X \bigvee_0^{u-1} S_{-i/N}(P^* \vee Q) - \operatorname*{dist}_X \bigvee_0^{u-1} S_{-i/N}(\mathbf{P}^* \vee Q)\right| < 26\delta/100 < \delta.$$

(v): We have $|\pi_1(P^* \vee Q) - \tilde{P}^*| < \rho_0/100$ and since $|P^* \vee Q - \boldsymbol{P}^* \vee \boldsymbol{Q}| < \rho_1$ and $|\tilde{P}^* - \tilde{\boldsymbol{P}}^*| < 11\rho_1/100$ we have $|\pi_1(\boldsymbol{P}^* \vee \boldsymbol{Q}) - \tilde{\boldsymbol{P}}^*| < 13\rho_0/100$. It follows that

$$|\pi_1(\mathbf{P}^* \vee \mathbf{Q}) - \tilde{\mathbf{P}}^*| < 13\rho_0/100 + 12\rho_1/100 < 25\rho_0/100$$

and that

$$|\pi_1(\mathbf{P}^* \vee Q) - \tilde{P}^*| < 38\rho_0/100.$$

Therefore

$$\begin{aligned} h(S_{1/N}, P^* \vee Q) &\geqq h(S_{1/N}, \mathbf{P}^* \vee Q) \\ &= h(S_{1/N}, \pi_1(P^* \vee Q) \vee Q) \\ &> h(S_{1/N}, \tilde{P}^* \vee Q) - \delta \\ &= h(S_{1/N}, P^* \vee Q) - \delta, \end{aligned}$$

which is the desired conclusion.

(vi): We have

$$\begin{aligned} \|\mathbf{p}^* - \psi_N \mathbf{p}^*\|_{X,1/N} &\leqq \|\mathbf{p}^* - p^*\|_{X,1/N} + \|p^* - \psi_N p^*\|_{X,1/N} + \| + \psi_N p^* - \psi_N \mathbf{p}^*\|_{X,1/N} \\ &\leqq 22\rho_1/100 + \xi/100 < 2\xi/200. \end{aligned}$$

It follows that $\|\mathbf{p}^* - \psi_N \mathbf{p}^*\|_{X,1/N} < 2\xi/100 + 11\rho_1/100 < 3\xi/100$ and we are finished.

Lemma 5. *Let $(S, P \vee Q \vee \tilde{P})$ be a process on $(X, \mathcal{F}, \mu)$ such that $(P \vee Q)_S = \mathcal{F}$, S_1 is ergodic, $(Q)_S$ has a Bernoulli complement in $(S_1, \mathcal{F})$ and $h(S_1) = h(S_1, (\tilde{P} \vee Q)_S)$.*

Then $(\forall \varepsilon > 0)$ there exists a partition $\tilde{P}_1$ such that $|\tilde{P} - \tilde{P}_1| < \varepsilon$, $\bar{d}_{Q,Q}[(S, \tilde{P}_1 \vee Q), (S, \tilde{P} \vee Q)] = 0$ and $P \subset_\varepsilon (\tilde{P}_1 \vee Q)_S$.

Proof. Let P^* and $\tilde{P}^*$ be partitions satisfying $P^* \supset P$, $\tilde{P}^* \supset \tilde{P}$, $(P^* \vee Q)_{S_1} = (P^* \vee Q)_S$, and $(\tilde{P}^* \vee Q)_{S_1} = (\tilde{P}^* \vee Q)_S = (\tilde{P} \vee Q)_S$. Let λ and $\tilde{\lambda}$ be the codes such that $\lambda(P^*) = P$ and $\tilde{\lambda}(\tilde{P}^*) = \tilde{P}$.

It will suffice to produce a partition $\tilde{P}_1^*$ such that $|\tilde{P}_1^* - \tilde{P}^*| < \varepsilon$, $\bar{d}_{Q,Q}[(S, \tilde{P}_1^* \vee Q), (S, \tilde{P}^* \vee Q)] = 0$, and $P^* \subset_\varepsilon (\tilde{P}_1^* \vee Q)_S$ for then $\tilde{\lambda}(\tilde{P}_1^*)$ will serve as the desired partition $\tilde{P}_1$. Let π_1 be a code on $\bigvee_{-N_1}^{N_1} S_i(P^* \vee Q)$ such that $|\pi_1(P^* \vee Q) - \tilde{P}^*| < \varepsilon/50$. Applying Lemma 4 we obtain a partition $P_0^* \subset (\tilde{P}^* \vee Q)_S$ such that $\bar{d}_{Q,Q}[(S, P_0^* \vee Q), (S, P^* \vee Q)] = 0$ and $|\pi_1(P_0^* \vee Q) - \tilde{P}^*| < 7\varepsilon/50$.

Now let π_2 be a code on $\bigvee_{-N_2}^{N_2} S_i(\tilde{P}^* \vee Q)$ such that $|\pi_2(\tilde{P}^* \vee Q) - P_0^*| < \varepsilon/7$. Since $(\tilde{P}^* \vee Q)_S \subset (P^* \vee Q)_S$, Theorem C implies that $(Q)_S$ has a Bernoulli complement in $(S_1, (\tilde{P}^* \vee Q)_S)$, so we apply Lemma 4 again to obtain a partition $\tilde{P}_0^* \subset (P_0^* \vee Q)_S$ such that

$$\bar{d}_{Q,Q}[(S, \tilde{P}_0^* \vee Q), (S, \tilde{P}^* \vee Q)] = 0,$$

$$|\pi_1(P_0^* \vee Q) - \tilde{P}_0^*| < 49\varepsilon/50,$$

and

$$|\pi_2(\tilde{P}_0^* \vee Q) - P_0^*| < \varepsilon.$$

Now since $(S, P^* \vee Q) \sim (S, P_0^* \vee Q)$, there is a partition $\tilde{P}^* \subset (P^* \vee Q)_S$ such that

$$\bar{d}_{Q,Q}[(S, \tilde{P}_1^* \vee Q), (S, \tilde{P}^* \vee Q)] = 0, \quad |\pi_1(P^* \vee Q) - \tilde{P}_1^*| < 49\varepsilon/50$$

and

$$|\pi_2(\tilde{P}_1^* \vee Q) - P^*| < \varepsilon.$$

Since $|\pi_1(P^* \vee Q) - \tilde{P}^*| < \varepsilon/50$ we see that $|\tilde{P}^* - \tilde{P}_1^*| < \varepsilon$ and we are done.

We are now able to prove the relative isomorphism theorem.

THEOREM 2. *Let $(\bar{S}, \bar{P} \vee \bar{Q})$ be a process on $(\bar{X}, \bar{\mathcal{F}}, \bar{\mu})$ such that $\bar{S}_1$ is ergodic, $(\bar{P} \vee \bar{Q})_{\bar{S}} = \bar{\mathcal{F}}$, and $(\bar{Q})_{\bar{S}}$ has a Bernoulli complement in $(\bar{S}_1, \bar{\mathcal{F}})$.*

Let $(S, P \vee Q)$ be a process on $(X, \mathcal{F}, \mu)$ satisfying the same description and such that $(S, Q) \sim (\bar{S}, \bar{Q})$ and $h(S_1) = h(\bar{S}_1)$.

Then S and $\bar{S}$ are isomorphic relative to their common factors (S, Q) and $(\bar{S}, \bar{Q})$.

PROOF. We will produce a partition $\tilde{P}$ of X such that

$$\bar{d}_{Q,\bar{Q}}[(S, \tilde{P} \vee Q), (\bar{S}, \bar{P} \vee \bar{Q})] = 0 \quad \text{and} \quad (\tilde{P} \vee Q)_S = \mathcal{F}.$$

Fix $\varepsilon_0 > 0$. By the Corollary to Theorem 1 and by the argument of Lemma 5 there is a partition $\tilde{P}_0$ of X such that $\bar{d}_{Q,\bar{Q}}[(S, \tilde{P}_0 \vee Q), (\bar{S}, \bar{P} \vee \bar{Q})] = 0$, and which is refined by a partition $\tilde{P}_0^*$ such that $(\tilde{P}_0^* \vee Q)_{S_1} = (\tilde{P}_0^* \vee Q)_S = (\tilde{P}_0 \vee Q)_S$ and such that $|\pi_0(\tilde{P}_0^* \vee Q) - P| < \varepsilon_0/2$ where π_0 is a code on $\bigvee_{-N_0}^{N_0} S_{-i}(\tilde{P}_0^* \vee Q)$.

We now iterate this construction as follows. For each $k = 1, 2, \cdots$ we choose $\varepsilon_k > 0$, $\varepsilon_k < \varepsilon_{k-1}/2$ so that any partition R satisfying $|R - \tilde{P}_{k-1}^*| < \varepsilon_k$ also satisfies $|\pi_{k-1}(R \vee Q) - \pi_{k-1}(\tilde{P}_{k-1}^* \vee Q)| < \varepsilon_{k-1}/2$. Then we apply Lemma 5 to $(S, P \vee Q \vee \tilde{P}_{k-1}^*)$ and $\varepsilon_k/2$ to obtain a partition $\tilde{P}_k^*$ such that $|\tilde{P}_{k-1}^* - \tilde{P}_k^*| < \varepsilon_k/2$, $\bar{d}_{Q,Q}[(S, \tilde{P}_k^* \vee Q), (S, \tilde{P}_{k-1}^* \vee Q)] = 0$, and for some code π_k on $\bigvee_{-N_k}^{N_k} S_i(\tilde{P}_k^* \vee Q)$, $|\pi_k(\tilde{P}_k^* \vee Q) - P| < \varepsilon_k/2$.

Then the partitions $\tilde{P}_k^*$ converge in the partition metric to a partition $\tilde{P}^*$ such that $\bar{d}_{Q,Q}[(S, \tilde{P}^* \vee Q), (S, \tilde{P}_0^* \vee Q)] = 0$ and for every $k = 0, 1, 2, \cdots$ $|\tilde{P}^* - \tilde{P}_k^*| < \varepsilon_k$ so that $|\pi_k(\tilde{P}^* \vee Q) - P| < \varepsilon_k$. Hence $(P \vee Q)_S \subset (\tilde{P}^* \vee Q)_S$. Now if $\tilde{\lambda}$ is the code such that $\tilde{\lambda}(\tilde{P}_0^*) = \tilde{P}_0$, then $\tilde{\lambda}(\tilde{P}^*)$ is the desired partition $\tilde{P}$.

An equivalent but perhaps more striking formulation of Theorem 2 is the following.

THEOREM 2′. *Let S be a flow on $(X, \mathcal{F}, \mu)$ of finite entropy with S_1 ergodic. Let $\mathcal{C} \subset \mathcal{F}$ be a factor of S such that $(S_1, \mathcal{C})$ has a Bernoulli complement in $(S_1, \mathcal{F})$. Then $(S, \mathcal{C})$ has a Bernoulli complement in $(S, \mathcal{F})$.*

PROOF. Let S' be a Bernoulli flow with $h(S_1') = h(S_1, \mathcal{F}) - h(S_1, \mathcal{C})$, and let

B' be a finite generator for S'. Let Q be a finite generator for $(S, \mathscr{C})$ and $P \vee Q$ a finite generator for $(S, \mathscr{F})$. If we set $(\bar{X}, \bar{\mathscr{F}}, \bar{\mu}) = (X' \times X,\ B' \times \mathscr{C},\ \mu' \times \mu)$ and $\bar{S} = S' \times S$, then, regarding B' and Q as partitions $\bar{B}$ and $\bar{Q}$ of $\bar{X}$, we see that $(\bar{S}, \bar{B} \vee \bar{Q})$ and $(S, P \vee Q)$ are processes of the sort to which Theorem 2 applies. Hence they are isomorphic via an isomorphism $\phi : \bar{X} \to X$ which carries $(\bar{S}, \bar{Q})$ onto (S, Q). Therefore $\mathscr{B} = \phi(\bar{\mathscr{B}})$ is the desired factor.

3. Consequences of the relative isomorphism theorem

THEOREM 3. *Let S be a flow on $(X, \mathscr{F}, \mu)$ with S_1 ergodic and with a factor $\mathscr{C} \subset \mathscr{F}$ such that $(S_1, \mathscr{C})$ has a Bernoulli complement in $(S_1, \mathscr{F})$. Let $\mathscr{A} \subset \mathscr{F}$ be a factor of $(S, \mathscr{F})$ containing $\mathscr{C}$. Then $(S, \mathscr{C})$ has a Bernoulli complement in $(S, \mathscr{A})$.*

PROOF. It follows from Theorems A, A′ and C that $(S_1, \mathscr{C})$ has a Bernoulli complement in $(S_1, \mathscr{A})$. Applying Theorem 2′ to the flow $(S, \mathscr{A})$ and the factor $(S, \mathscr{C})$ we obtain the result.

THEOREM 4. *For each $i = 1, 2, 3, \cdots$ let S^i be a flow on $(X_i, \mathscr{F}_i, \mu_i)$ with S^i_1 ergodic. Let P_i and Q_i be partitions of X_i such that $(S^i, (Q_i)_{S^i})$ has a Bernoulli complement in $(S^i, (P_i \vee Q_i)_{S^i})$. Suppose that a process $(S, P \vee Q)$ is the $\bar{d}$-limit of the $(S^i, P_i \vee Q_i)$ and each (S^i, Q_i) is isomorphic to (S, Q). Suppose also that $\{h(S^i, (P_i \vee Q_i)_{S^i})\}_{i=1}^{\infty}$ is a bounded sequence. (This will be the case, for example, if $(\forall i)$ $(P_i \vee Q_i)_{S^i_1} = (P_i \vee Q_i)_{S^i}$.) Then $(S, (Q)_S)$ has a Bernoulli complement in $(P \vee Q)_S$.*

PROOF. Let $(\hat{S}, \hat{P} \vee \hat{Q})$ be a process on $(\hat{X}, \hat{\mathscr{F}}, \hat{\mu})$ such that $(\hat{S}, \hat{Q})$ is isomorphic to (S, Q), $(\hat{S}, (\hat{Q})_{\hat{S}})$ has a Bernoulli complement in $(\hat{S}, (\hat{P} \vee \hat{Q})_{\hat{S}})$, and $h(\hat{S}) \geqq \sup_i \{h(S^i, (P_i \vee Q_i)_{S^i})\}_{i=1}^{\infty}$. Choose $\{\varepsilon_k > 0\}_{k=1}^{\infty}$ so that $\Sigma_{k=1}^{\infty} \varepsilon_k < \infty$. Now choose $\delta_k < \varepsilon_k$ and a subsequence $\{i_k\}_{k=1}^{\infty} \subset \mathbf{N}$ so that

$$(\forall k) \quad \bar{d}_{Q_{i,k}, Q_{i,k+1}}[(S^{i_k}, P_{i_k} \vee Q_{i_k}), (S^{i_{k+1}}, P_{i_{k+1}} \vee Q_{i_{k+1}})] < \delta_k .$$

By Remark 3 following Theorem 1, we may choose δ_k so that $(\forall k)(\exists \hat{P}_k)$ with $\bar{d}_{\hat{Q}, Q_{i_k}}[(\hat{S}, \hat{P}_k \vee \hat{Q}), (S^{i_k}, P_{i_k} \vee Q_{i_k})] = 0$ and $|\hat{P}_k - \hat{P}_{k+1}| < \varepsilon_k$. Then $\hat{P}_0 = \lim_k \hat{P}_k$ satisfies $\bar{d}_{\hat{Q}, Q}[(\hat{S}, \hat{P}_0 \vee \hat{Q}),\ (S, P \vee Q)] = 0$ and by Theorem 3, $(\hat{S}, (\hat{Q})_{\hat{S}})$ has a Bernoulli complement in $(\hat{S}, (\hat{P}_0 \vee \hat{Q})_{\hat{S}})$, which establishes the result.

THEOREM 5. *Let S be a flow on $(X, \mathscr{F}, \mu)$ and $\mathscr{C} \subset \mathscr{F}$ a factor of S_1 such that $(S_1, \mathscr{C})$ is an ergodic automorphism of zero entropy with a Bernoulli complement in $(S_1, \mathscr{F})$.*

Then (a) *$(S, \mathscr{F})$ is isomorphic to the direct product of a Bernoulli flow and an* *ergodic flow of zero entropy, and* (b) *for all factors* $\mathscr{A} \subset \mathscr{F}$ *of* S, $(S, \mathscr{A})$ *is also a direct product of this type.*

PROOF. (a) It follows from Theorem C that $\mathscr{C}$ is the Pinsker algebra (the maximal factor of zero entropy) of $(S_1, \mathscr{F})$. But $(\forall t \in R)$, S_t provides an isomorphism of S_1 with itself, so that $S_t\mathscr{C} = \mathscr{C}$. In other words, $\mathscr{C}$ is necessarily flow-invariant. Theorem 2′ then implies that $(S, \mathscr{C})$ has a Bernoulli complement in $(S, \mathscr{F})$.

(b) If $\mathscr{A} \subset \mathscr{F}$ is a factor of S, then Theorem F (a) implies that $(S_1, \mathscr{A})$ is the direct product of a Bernoulli shift and an ergodic automorphism of zero entropy. Now part (a) applied to $(S, \mathscr{A})$ yields (b).

THEOREM 6. *For each* $i = 1, 2, 3, \cdots$ *let* S^i *be a flow on* $(X_i, \mathscr{F}_i, \mu_i)$ *isomorphic to the direct product of a Bernoulli flow and an ergodic flow of zero entropy. Let* P_i *be a partition of* X_i *and suppose that the processes* (S^i, P_i) *converge in the* $\bar{d}$*-metric to a process* (S, P) *on* $(X, \mathscr{F}, \mu)$ *where* $(P)_{S_1} = \mathscr{F}$. *Then* S *is the direct product of a Bernoulli flow and an ergodic flow of zero entropy.*

PROOF. The processes (S^i_1, P_i) converge in $\bar{d}$ to (S_1, P) so Theorem F (b) implies that S_1 is isomorphic to the direct product of a Bernoulli shift and an ergodic automorphism of zero entropy. Theorem 5 now yields the result.

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